



On Sampling for Surfaces Reconstruction

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Abstract

In CAD and reverse engineering, triangulation, i.e. C^0 interpolation, of scattered sampled points should has the property of shape-preserving for precisely reconstruction of original surface. This depends much more on sampling. It is well known that over-sampling or under-sampling either increases computing consumption in triangulation or cannot get the correct reconstruction. In this paper, the local structure of 3D curve is firstly analyzed in frequent domain with Fourier transformation. And then the sampling frequency based on Shannon theorem is discussed. Subsequently, generalizing it to the surface case, we present in particularly a sampling method for 3D surfaces. The results indicate that, with the method, dense enough triangulations can be obtained so as to avoid over- and under-sampling.

point-set in surface reconstruction and model simplification. It is based on Voronoi cell geometry. In [2], the stochastic sampling method (SSM) was proposed to numerically generate sample points on implicit surfaces. In [3], applications of SSM to various complicated implicit surfaces were demonstrated. However, in the works of SSM, it is applicable to implicit surfaces.

This paper focuses on that how to sample can result in shape-preserving triangulations when facing existent surfaces, as well as avoiding superfluous and insufficient samples. To begin with analysis of local property of 3D curves, we probe into sampling density of curve case. After that, generalizing it to surface case, a method for surface sampling is built, which is theoretically and instructively significant. Finally, the method is tested and verified through detailed analyses to some examples.

1. Introduction

In CAD and reverse engineering, it is often to precisely reconstruct an original surface from the scattered points sampled. Usually, reconstruction starts with the triangulation, i.e., C^0 interpolation, of the points. The triangulation, named shape-preserving triangulation, must be consistent to and correctly represent the original surface. In order to get shape-preserving triangulation, how to sample plays an important role. Sampling too densely, i.e., over-sampling, will increase computing consumption; sampling too sparsely, i.e., under-sampling, will lead to inconsistent triangulation. In [1], a guide was proposed to detect under- and over-sampling of a sampled

2. Sampling 3D curves

2.1. Local structure of a 3D curve

Let $\Gamma: r(s)$ be a space curve with continuity of third derivative. Considering the moving 3-prism and Frenet-frame of a point P along a curve, α , β and γ form the basic vectors. They are Γ 's invariant vectors which meeting Frenet-Serret formulation:

$$\begin{bmatrix} \dot{\alpha} \\ \dot{\beta} \\ \dot{\gamma} \end{bmatrix} = \begin{bmatrix} & \kappa & \\ -\kappa & & \tau \\ & -\tau & \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix},$$

where κ , τ is absolute curvature and torsion at P

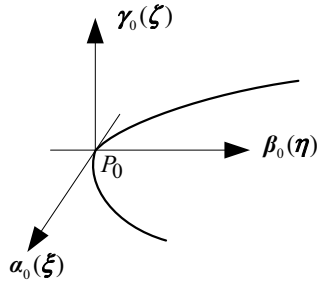


Figure 1. The parabola in osculating plane as an approximation of a segmental curve neighboring the point.

respectively.

Let α_0 , β_0 , γ_0 , κ_0 , τ_0 and κ_0 be the values at P_0 (where $s=s_0$). According to the local theory of space curve [4], when taking $[P_0; \alpha_0, \beta_0, \gamma_0]$ as a new coordinate system, a curve $\bar{\Gamma}$ approximate to Γ nearing P_0 is:

$$\bar{\Gamma}: \begin{cases} \xi = s \\ \eta = \frac{1}{2} \kappa_0 s^2 \\ \zeta = \frac{1}{6} \kappa_0 \tau_0 s^3 \end{cases}.$$

Obviously, $\bar{\Gamma}$ and Γ have the same curvature and torsion at P_0 . Since curvature plays a key role, we consider the projection of $\bar{\Gamma}$ onto the osculating plane $\zeta=0$:

$$\begin{cases} \xi = s \\ \eta = \frac{1}{2} \kappa_0 s^2 \end{cases} \quad (2.1)$$

It is a parabola. Let $r_0 = \frac{1}{\kappa_0}$ be the curvature radius at P_0 ,

then (2.1) can be described as (see Figure 1):

$$\Gamma_p: \eta = \frac{1}{2r_0} \xi^2 \quad (2.2)$$

A segment of Γ_p will be taken as the approximation of Γ around P_0 .

2.2. Sampling frequency for 3D curves

The function $\Pi(x)$ is defined as:

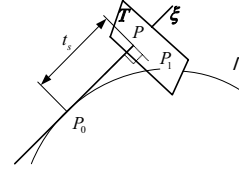


Figure 2. Tangent sampling step.

$$\Pi(x) = \begin{cases} 0, & |x| > \frac{1}{2} \\ \frac{1}{2}, & |x| = \frac{1}{2} \\ 1, & |x| < \frac{1}{2} \end{cases}.$$

The Fourier transformation of $\Pi(x)$ is:

$$\Pi(\omega) = \int_{-\infty}^{\infty} \Pi(x) e^{-i2\pi x \omega} dx = \frac{\sin \pi \omega}{\pi \omega}. \quad (2.3)$$

A double-pulse function is defined as:

$$\Pi(x) = \frac{1}{2} (\delta(x + \frac{1}{2}) + \delta(x - \frac{1}{2})).$$

Its Fourier transformation is:

$$\Pi(\omega) = \cos(\pi \omega) \quad (2.4)$$

As shown in Figure 2, tangent sampling step is defined.

Definition: sampling from P_0 on Γ , the next point P_1 is obtained as: get tangent line ξ at P_0 , and then find a plane $T \perp \xi$ along the sampling direction. The perpendicular foot is P . P_1 is the intersection point of T and Γ . The length of line segment P_0P is named as tangent sampling step t_s from P_0 to P_1 .

Theorem: when sampling a 3D curve, if tangent sampling step satisfy:

$$t_s \leq 0.7 \rho r_0, \rho \in (0, 1], \quad (2.5)$$

then the curve can be reconstructed from the sampling points, and call that this kind of sample have ρ -precision.

Proof. Taking a segment of Γ_p within interval $\xi \in [-\rho r_0, \rho r_0]$ along ξ axis (see Figure 1), where $\rho \in (0, 1]$. For convenience, we make a coordinate transformation to this segment (as shown in Figure 3). Now consider $f(x)$ as:

$$f(x) = -\frac{1}{2r_0} (x^2 - (\rho r_0)^2) \Pi(\frac{x}{2\rho r_0}).$$

The first and second derivatives of are:

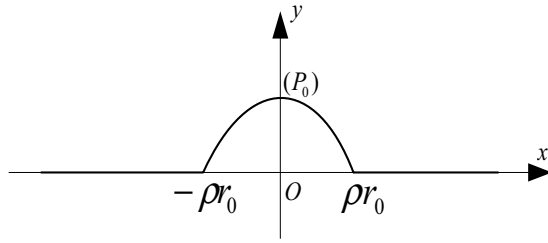


Figure 3. A segment of parabola Γ_p .

$$\dot{f}(x) = -\frac{1}{r_0} x \Pi\left(\frac{x}{2\rho r_0}\right)$$

and

$$\ddot{f}(x) = -\frac{1}{r_0} \left(\Pi\left(\frac{x}{2\rho r_0}\right) - (\delta(x + \rho r_0) + \delta(x - \rho r_0)) \right)$$

respectively. From (2.3) and (2.4) it is got that:

$$\ddot{f}(\omega) = -\frac{1}{r_0} \left(\frac{\sin 2\rho r_0 \pi \omega}{\pi \omega} - 2\rho r_0 \cos 2\rho r_0 \pi \omega \right).$$

According to the property of Fourier transformation, it is found that:

$$f(\omega) = \frac{\ddot{f}(\omega)}{(i2\pi\omega)^2} = \frac{1}{4r_0} \left(\frac{\sin 2\rho r_0 \pi \omega}{\pi^3 \omega^3} - 2\rho r_0 \cos 2\rho r_0 \pi \omega \right),$$

as shown in Figure 4.

Take ω as cut-off frequency ω_c when $\omega > 0$ is the smallest one of those meeting $f(\omega) = 0$, then:

$$\omega_c = \frac{4.493}{2\pi\rho r_0}.$$

According to Shannon sampling theorem, sampling frequency satisfies $f_s \geq 2\omega_c$. Thereby sampling step satisfies $t_s = \frac{1}{f_s} \leq \frac{1}{2\omega_c} = 0.7\rho r_0$.

From the analysis above we can get the theorem of 3D curve sampling as:

Remarks:

(1) Analysis starts with Frenet-frame of a point which doesn't exist at tarry points (where $\dot{\mathbf{r}} \times \ddot{\mathbf{r}} = \mathbf{0}$). However, the curve that consists of only tarry points is a line. If a curve with continuous curvature doesn't include any segment of line, the tarry points on it are isolated. From (2.2), it

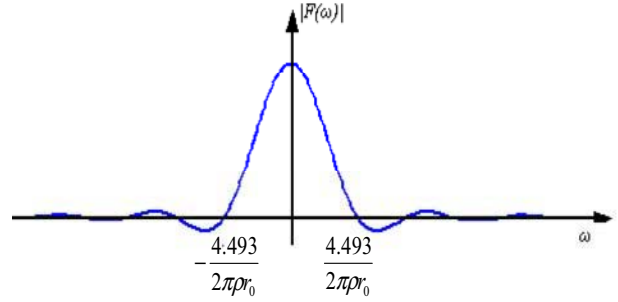


Figure 4. Fourier transform $f(\omega)$ of $f(x)$.

doesn't influence the result of analysis provided that $\ddot{\mathbf{r}} \neq \mathbf{0}$;

(2) At an inflection point (where $\kappa=0$), $r=\infty$, i.e., the curvature circle there degenerates to a line. In this case only the inflection point is sampled there; if $\kappa=0$ anywhere for a curve, it is obviously a line. Only two end points are sampled for a line.

(3) In (2.5), ρ is changeable. Thus, sampling can proceed with a constant of t_s if needed.

2.3: Method for sampling 3D curves

Given the theorem, we keep on to discuss the method for sampling 3D curves. The following method 1 shows how to get ρ -precision samples nearing P_0 on a curve Γ .

Method 1:

Along the tangent line of P_0 , two points P'_{0l} and P'_{0r} are got at the distance t_s from P_0 . Through P'_{0l} and P'_{0r} there exists two planes perpendicular to P_0 's tangent line. These two planes and Γ intersect at P'_0 and P''_0 respectively where P'_0 and P''_0 are the sampling points nearing P_0 with ρ -precision.

The following method 2 shows how to sample a 3D curve Γ .

Method 2:

Step 1. Sampling all the inflection points and end points of Γ (only inflection points for a close curve).

Step 2. The points got in step 1 divide Γ into several segments, and for each segment:

- a. From the point P_m with the maximum curvature, getting P_m^l and P_m^r with method 1;
- b. From P_m^l getting P_m^{l+1} in direction $\vec{P_m P_m^l}$ with method 1. Then getting P_m^{l+2} from P_m^{l+1} ... similarly. Thus, a series of points are got from P_m^l . Likewise, a series of points P_m^{r+i} are got from P_m^r ;
- c. Stopping step b once beyond the boundary.

Step 3. Repeating step 2 until all segments are sampled.

3. Method for sampling 3D surfaces

It appears to be complicated for the surface case. We know from the knowledge of differential geometry that a surface without umbilical points can be parameterized to have lines of curvature as its parameter network because these lines form orthogonal networks. In the light of this, the following method 3 for sampling a surface is built.

Method 3:

Step 1. Dividing the surface $\mathcal{S}$ to be sampled into several sub-surfaces $\mathcal{S}_i$ with continuous curvature;

Step 2. For a $\mathcal{S}_i$, sampling its boundary curves using method 2;

Step 3. As to the points got in step 2, their lines of curvature divide $\mathcal{S}_i$ into several patches. Then sampling the boundaries of these patches;

Step 4. For each patch got in step 3:

- a. Getting all parabolic points (where one of principal curvatures is 0);
- b. For a parabolic point P_i , let $\kappa_{i,1}$ and $\kappa_{i,2}$ be its principal curvatures and $\kappa_{i,1} = 0$, then sampling the line of curvature of P_i in direction of $\kappa_{i,1}$;
- c. The patch is divided into several sub-patches by lines of curvature got in step b. For each sub-patch $\mathcal{S}_p$:
 - c.1. Let $P_m \in \mathcal{S}_p$ with principal curvatures $\kappa_{m,1}$ and $\kappa_{m,2}$ satisfy that $\max(\kappa_{m,1}, \kappa_{m,2}) = \max_{P_i \in \mathcal{S}_p} (\max(\kappa_{i,1}, \kappa_{i,2}))$. Two lines of curvature of P_m in $\mathcal{S}_p$ are sampled. Sampling will be stopped when beyond the boundary;
 - c.2. Two lines of curvature in c.1 divided $\mathcal{S}_p$ into 4

partitions. Sampling each one as c.1;

Sampling each $\mathcal{S}_i$ as step 2~4 until all $\mathcal{S}_i$ s of $\mathcal{S}$ are sampled.

Method 3 is in fact a divide-and-conquer sampling procedure. It can be carried out recursively.

Some remarks for method 3:

- (1) If some $\mathcal{S}_i$ is parabolic (means all points on it are parabolic points), stop sampling after step 3;
- (2) If P_m is an umbilical point, take arbitrarily two

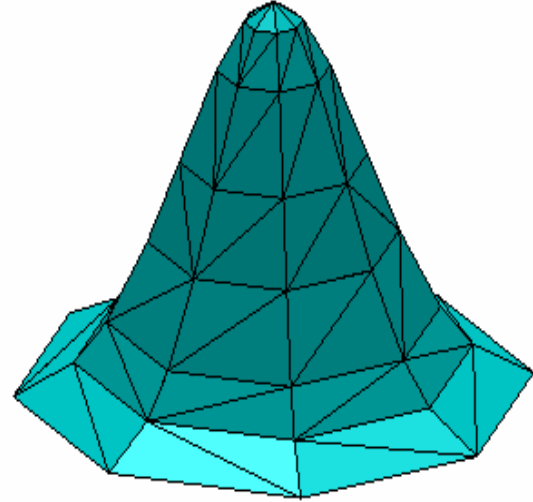


Figure 5. Dense enough sampling to the example 1.

orthogonal principal direction in c.1;

- (3) If each point $P_i \in \mathcal{S}_p$ has the same $\max(\kappa_{i,1}, \kappa_{i,2})$, a point at the center of $\mathcal{S}_p$ can be taken as P_m .

4. Experiments

Before discussing the experiments, let's have a brief review of optimal criteria about triangulations. Smoothness criterion [5] and space shape optimal criterion [6] can be applied to close surfaces successfully, but it is necessary to build pseudo boundary for open surfaces. Criterion of minimal absolute mean curvature [7], criterion of maximum scalar product of normal vectors [8] or ABN criterion and JND criterion [9] can be easily applied to open surfaces. In this paper, another criterion is employed. The following criterion is similar to the criterion of minimal absolute mean curvature. For each quadrilateral we define another estimation of mean

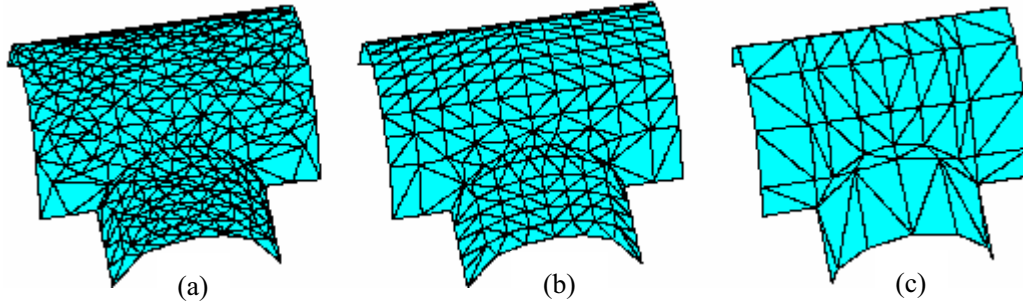


Figure 6. Results of example 2:(a)over-sampling.(b)sparser but still over sampling.(c)dense enough sampling.

curvature:

$$\kappa = \frac{\|n_1 - n_2\|}{\|C_1 - C_2\|}$$

as its cost function, where n_1 and C_1 , C_2 and n_2 are the unit normals and centroids of its two triangles respectively. One of the two triangulations corresponding to the quadrilateral with minimal κ will be taken as the optimal one.

The first Example is the triangulation of a part of $\mathcal{Z}: f(x, y) = \exp(-8\frac{1}{4}((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2))$. Unlike [Alboul ,2000] in which 641 points is sampled with uniform steps in x and y direction, all the 57 points here are sampled with method 3 under $\rho \approx 1$.

Finally, another example is given in detail. Let $\mathcal{Z}$ consists of: $\mathcal{Z}_{p_1} = \{(x, y, z): z^2 + x^2 = 0.6^2, z \geq 0, -2 \leq y < 0\}$ and $\mathcal{Z}_{p_2} = \{(x, y, z): z^2 + y^2 = 1, z \geq 0, -1 \leq x < 1\}$. A point-set of very dense sampling to $\mathcal{Z}$ includes 304 points with 42 boundary points. The result of triangulation is shown in Figure 7 (a). It is observed that too many spurious “wiggles” exist in the result. Apparently, it doesn’t reflect the truth of the original surface.

Now, let’s change the sampling density. We sample $\mathcal{Z}$ uniformly as to X - Y plane and a bit denser at the intersection. This is also the common method used in practice. The sample includes 178 points with 50 boundary points. The result of triangulation can be seen in Figure 7 (b). Clearly, this result is better than (a).

Because $\mathcal{Z}_{p_1}$ and $\mathcal{Z}_{p_2}$ are cylinders, i.e. every point on them has a principal curvature as 0. Although sampling in

Figure 7 (b) satisfies the sampling theorem, according to method 3, it is still too dense. Re-sampling it in accordance with method 3 with tangent step $t_s = 0.7r_0$ ($\rho = 1$), we get the result as shown in Figure 7 (c). It includes only 68 points with 30 boundary points, but it represents the real original surface very well.

5. Conclusions and future work

An instructive sampling method is introduced in this paper for surface reconstruction and some examples analyzed in detail. Preliminary results show that over-sampling not only increases the computing consumption, but also may lead to triangulation inconsistent to the original surface. Although it is impossible to calculate sampling steps for a surface without its concrete expression, application of this method can guide us to get dense enough sampling for surface reconstruction so as to simplify triangulation and avoid superfluous and insufficient sampling. Meanwhile, it is apparent that it is easy to apply to a surface with known expression.

The future study may lie mainly in:

- 1) Although many experiments indicate that triangulation with the optimal criterion proposed in this paper is of shape-preserving under the sampling method, it is not proved theoretically. What’s the sampling requirement for this criterion, especially at the intersection of two or more surfaces?
- 2) For the original surface with unknown expression how to estimate the main curvatures so as to determine tangent sampling step?

3) For a sampled point-set how to apply the proposed method to confirming where is over-sampling and where is under-sampling? It is very significant.

6. References

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